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COMPARISON OF PRE-SMOOTHING, EXPONENTIAL SMOOTHING, AND KALMAN FILTERING
APPLIED TO SEVERAL METHODS OF TARGET MOTION ANALYSIS

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13. ABSTRACT

Comparisons are made of the bias and standard deviation in the miss distance produced by random bearing errors when pre-smoothing, exponential smoothing, and Kalman filtering are used in several methods of target motion analysis. All three techniques, alone and in combination, are used to estimate the bearing and bearing rate that are needed for two new TMA methods. The effects of pre-smoothing on the CHURN TMA also are included in the comparison.

Good results are obtained with the two new TMA methods when the bearing and bearing rate are obtained by (a) exponential smoothing with no pre-smoothing, (b) exponential smoothing with moderate pre-smoothing, (c) maximum pre-smoothing (two groups), and (d) Kalman filtering with no pre-smoothing. Significantly larger biases and somewhat smaller standard deviations are obtained with CHURN when optimal pre-smoothing is used.

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COMPARISON OF PRE-SMOOTHING, EXPONENTIAL SMOOTHING, AND KALMAN FILTERING APPLIED TO SEVERAL METHODS OF TARGET MOTION ANALYSIS

INTRODUCTION

Random bearing errors may produce biases, as well as variations, in the target motion parameters. In reference (a), which contains references to earlier reports on this subject, it was reported that random bearing errors produce large biases and small variances with the CHURN TMA, while they produce small biases and large variances with two new TMA methods that were developed for controlling the MK 48 torpedo.

Various modifications to the tracking procedures and to the analysis of the bearing data have been proposed and used, in an effort to reduce the adverse effects of random bearing errors. Among these procedures are the following:

(a) Increase the duration T_1 of the tracking legs, and hence, the length of the total tracking interval.

(b) Increase the frequency of taking bearing observations, that is, decrease the interval Δt between observations.

(c) Pre-smooth the bearing observations before computing the target motion parameters; that is, group the bearing observations, average, and enter the averages into the estimators from which the target parameters are computed.

(d) Apply "exponential smoothing" directly in the computation of the parameters. This procedure is an "updating" procedure, rather than a "smoothing" procedure. It is a means of decreasing the weight given old data relative to new data.

(e) Use an estimator that minimizes an appropriate measure of the errors in an observed or computed quantity. The CHURN TMA is obtained from such a minimization. An

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alternative procedure is one that usually is called
Kalman filtering, which is essentially a linear, adaptive estimator that minimizes the expected value of
the square of the norm of an error vector.

Procedure (a) is a brute force method that is almost certain to improve
the accuracy of the estimates, provided the target is cooperative enough
to remain on a nearly linear path during the enlarged tracking interval.
Procedure (b) is an attempt to gain from the decrease in the variance of
a mean by the factor n^{-1} as the number n of __independent__ observations
increases. Procedure (c) uses the principle of procedure (b) in opposing
directions; by averaging in advance, the variance of each entry is reduced, at the cost of reducing the number of entries. Procedure (d) is
a common method of updating and limiting the effective time (the smoothing time) over which the computations extend; a more accurate name is
exponential weighting. Procedure (e) is a method of obtaining estimators
for the parameters that will achieve a desirable optimization.

These procedures are used, alone and in combinations, in various TMA
methods, and have been made an integral part of some of them. For example, CHURN is a least squares estimator that uses frequent bearing
observations and pre-smoothing, with a complicated rule for determining
the number of observations to average. Exponential weighting has been
used as a method of updating in many fire control systems, including
those used by the U. S. Navy for surface and anti-aircraft gunnery. We
have used it in our TMA estimators.

The effects of procedures (a) and (b) on CHURN and on our new TMA
methods were reported in reference (b). Procedure (a) reduces significantly the large biases in CHURN, approximately as T_1^{-1}. It has a
negligible effect on our new TMA methods, which produce small biases
with even small tracking legs. Procedure (b) has a negligible effect
on the bias in the CHURN solution, and reduces the variance (which is
small for all reasonable values of Δt) approximately as Δt^{-1}. Procedure (b) has a negligible effect on our new TMA methods. These results

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were obtained under the assumption that the random errors in the bearing observations are independent. The effects achieved by decreasing Δt will be less with correlation than they are with independent observations, and the error reductions are not large with the assumption of independence.

The main purpose of this memorandum is to report some preliminary results obtained in our study of the effects of pre-smoothing, exponential weighting, and the use of Kalman filtering on our new methods of target motion analysis and on CHURN. The results are obtained under the assumption that the random bearing errors are independently and identically distributed with a normal distribution having mean zero. Also, the analysis is limited to a small number of linear courses.

The restriction to zero means and linear courses is made to prevent the large errors that are obtained from bearing biases and target maneuvers from dominating the errors in the parameters and obscuring the effects being studied. The errors from these other sources are discussed in references (a) and (b), which, however, do not include the effects of biases and non-linear courses on Kalman filtering. The assumption of independence is made for simplicity. Since the assumption may have a significant effect on part of the comparisons, some of the computations will be repeated with an exponential autocorrelation function.

METHOD OF ANALYSIS

The method of analysis is random simulation to compute the miss distance described in reference (a). We simulate the motions of the target and tracking submarines, compute the true bearings, and add random bearing errors that are chosen randomly from a normal distribution having mean zero and standard deviation σ_b. The simulated bearings then are used to compute target parameters, using the estimators that characterize the particular TMA. The target parameters are used to compute the lead angle for interception of the target by the guide point, assumed here to be the same as the laminar point. The miss distance w normal to the relative motion line is computed for each simulated run. From a number of runs we compute the mean miss distance $\bar{w}$ and the standard deviation σ_w from the mean.

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The relative effects of the bias $\bar{w}$ and standard deviation σ_w depend on the application. The net bias (from all sources) has a greater relative effect on the acquisition probability than the total variance. When the net bias in the miss distance is large in magnitude, the optimal variance usually is not zero. Hence, there is no acceptable method of combining $\bar{w}$ and σ_w from the random bearing errors into a single measure. In general, we want $\bar{w}$ to be close to zero and σ_w^2 small enough to avoid dominating the total variance from all sources.

A difficulty was encountered in applying the acceptance test used with our TMA methods. The test seldom rejects the solution based on three legs when σ_b is small, say less than 0.5 degrees, but rejects many solutions when $\sigma_b = 1.0$ degree. If we allow the tracking to proceed to four, and perhaps more, legs to get a solution that is accepted by the test, the engagement geometry is changed and the comparison may be influenced by this change. If we remove the acceptance test to obtain the same engagement geometry for all runs, we are using some runs that would normally not be used in practice.

A compromise procedure was adopted. The acceptance test was applied and the run was accepted (rejected) for this analysis as the acceptance test was positive (negative) for the first three legs. If the test was negative, the run was terminated. Enough runs were made to obtain 30 accepted runs, since previous tests had shown that approximately 30 runs are needed to obtain accurate estimates of $\bar{w}$ and σ_w. The number of rejected runs is recorded. The restriction of the analysis to accepted runs has the effect of reducing the effect of σ_b on $\bar{w}$ and σ_w, since it would be expected that the magnitude of the average miss distance for the rejected runs would be larger than that for the accepted runs. The CHURN computations are made for the 30 accepted runs to put them on a comparable basis.

EFFECTS OF PRE-SMOOTHING

Effects of pre-smoothing on our TMA methods and on CHURN are shown in Table I for target courses of 90 and -45 degrees and $\sigma_b = 0.7$ degrees.

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The tracking legs are 100 seconds and the time step Δt is 2 seconds. Hence, with the group sizes of 1, 5, 10, and 25 the number of groups on each leg are 50, 10, 5, and 2 respectively. Thus, we consider the extreme of no pre-smoothing at one end and that of a split into two halves at the other end. As before, TMA No. 1 and TMA No. 2 are our new TMA methods with $\hat{U}_r = 0$ and the computed $\hat{U}_r$ respectively.

The main effect of pre-smoothing on our TMA methods is to reduce the standard deviation σ_w from a value near 1000 yards to 400 - 500 yards with two groups. The effect on $\bar{w}$ is not uniform and perhaps may not be accurately displayed by a sample of 30 runs when $\sigma_b = 0.7$ degrees.

The main effect of pre-smoothing on the CHURN TMA is to reduce the magnitude of the bias and increase the standard deviation. The optimal amount of pre-smoothing will depend on the effects of these changes on the acquisition probability, as discussed above. From the results shown in Table I it seems likely that the optimal value is near the case of 5 groups of 10 each for the assumptions we have used.

The large errors when TMA No. 1 and TMA No. 2 are used with no pre-smoothing raises the question of the source of these errors. Also, it would not be expected that pre-smoothing would have a large effect on the estimate of bearing rate. A study of these questions shows that the errors in the final bearings on the tracking legs may produce a large error in the miss distance when σ_b exceeds 0.5 degrees. Since a significant part of the variance σ_w^2 comes from this source, σ_w^2 is reduced by using the average of the last group for the final bearing.

The computations were repeated with the final bearing, as well as the bearing rate, computed by exponentially-weighted least squares. All other conditions are the same as those for Table I, including the value $\sigma_b = 0.7$ degrees. The results are shown in Table II. The results for TMA No. 1 and TMA No. 2 with no pre-smoothing are improved significantly by this change. Also, pre-smoothing has very little effect, except to increase σ_w as the group size increases to 10 and then to decrease it to a value approximately equal to the original value as the group size goes

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to 25. An additional gain from this procedure is a large reduction in the number of rejected runs for small group sizes.

It would appear from this analysis that we should compute $\hat{B}$ by exponentially weighted least squares, rather than use the last observed bearing, which disagrees with the conclusion we had reached in reference (b). If we estimate only the bearing rate $\dot{B}$ by least squares, the variance in the error of our estimate $\hat{\dot{B}}$ from its mean is reduced significantly (by a factor of nearly 5) from the corresponding variance when both parameters are estimated by least squares. The expected gain from this reduction is not realized because the error in $\hat{\dot{B}}$ from the random bearing errors does not produce the dominant error in the miss distance; the errors in the bearing estimate $\hat{B}$ have a larger effect than the errors in the estimate of the bearing rate when $\hat{B}$ is obtained from the last bearing observation.

The values of $\overline{w}$ and σ_w for CHURN are computed under the same conditions in Tables I and II. The slight differences in the values listed in the two tables, except for the group size 25, occur from the differences in the number of rejected runs; the 30 runs over which $\overline{w}$ and σ_w are computed are not the same in the two tables, except for the largest group size.

BEARING AND BEARING-RATE ESTIMATES FROM A KALMAN FILTER

Another method of computing the bearing and bearing rate needed in TMA No. 1 and TMA No. 2--and these are the only estimates needed from the bearing data to use these two TMA methods--is to obtain them from the least squares estimator called the Kalman filter. Discussions of the Kalman filter can be found in references (c) and (d) and in documents referenced there. We extract the special case that will be used here.

We want to estimate the bearing B and the bearing rate f at the end of m observations of the noisy bearing. We assume that the bearing rate can be approximated by a constant and the bearing by a linear function of time. Let

$$\hat{X}(m) = \begin{bmatrix} \hat{f}_m \\ \hat{B}_m \end{bmatrix}, \quad X^*(m) = \begin{bmatrix} f_m^* \\ B_m^* \end{bmatrix} \tag{1}$$

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where $\hat{f}_m$, $\hat{B}_m$ are predicted values for the mth stage from the $(m-1)^{st}$ stage, and f_m^*, B_m^* are the "best" estimates when the observed bearing B_m is used to revise the predicted values.

The Kalman estimator $X^*(m)$ is the following:

$$X^*(m) = \hat{X}(m) + K(m)\ (B_m - \hat{B}_m) \qquad (2)$$

where

$$\hat{X}(m) = A(m-1)\ X^*(m-1)\ , \qquad (3)$$

$$A(m) = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \text{ for all } m\ , \qquad (4)$$

and $K(m)$ is a column vector that is obtained by imposing the condition that the expected value of the square of the norm of $X(m) - \hat{X}(m)$ be a minimum. In writing equations (2) (3) and (4) we have omitted terms in the general theory that are not relevant here, and have specified that the matrix $A(m-1)$ in the prediction equation is constant, as shown in equation (4).

The $K(m)$ vector often is called the Kalman filter. It supplies the weights that are to be applied to the error, $B_m - \hat{B}_m$, in the correction (2). For our application it is obtained as follows:

$$K(m) = P(m|m-1)\ M' \left[M\ P(m|m-1)\ M' + \sigma_b^2 \right]^{-1} \qquad (5)$$

$$P(m|m-1) = A\ P(m-1|m-1)\ A' \qquad (6)$$

$$P(m|m) = [I - K(m)M]\ P(m|m-1) \qquad (7)$$

where

$$M = \begin{bmatrix} 0 & 1 \end{bmatrix}\ ,\quad A = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix},\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (8)$$

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and $P(m|k)$ is the covariance matrix for f and B at index m, given k observations have been made, $k=m-1,\ m$. We have written equation (5) in the special form that applies when the observations are limited to the bearing with a standard deviation σ_b. Also, we have omitted a random input in equation (6).

We compute $K(m)$ iteratively, starting with estimates for the covariance matrix. For example, start with

$$P(0|0) = \begin{bmatrix} \hat{\sigma}_{ff} & \hat{\sigma}_{fB} \\ \hat{\sigma}_{Bf} & \hat{\sigma}_{BB} \end{bmatrix}, \tag{9}$$

compute $P(1|0)$ from equation (6), $K(1)$ from equation (5), and $P(1|1)$ from equation (7). Then repeat. For convenience put

$$P(m|m) = \begin{bmatrix} a_m & b_m \\ c_m & d_m \end{bmatrix}, \quad P(m|m-1) = \begin{bmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{bmatrix}$$

Then equation (5) reduces to

$$K(m) = h_m \begin{bmatrix} \beta_m \\ \delta_m \end{bmatrix}, \quad h_m = 1/(\delta_m + \sigma_b^{\,2}) \tag{5'}$$

and equations (6) and (7) become

$$\begin{bmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} a_{m-1} & b_{m-1} \\ c_{m-1} & d_{m-1} \end{bmatrix} \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \tag{6'}$$

$$\begin{bmatrix} a_m & b_m \\ c_m & d_m \end{bmatrix} = \begin{bmatrix} 1 & -\beta_m h_m \\ 0 & \sigma_b^{\,2} h_m \end{bmatrix} \begin{bmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{bmatrix} \tag{7'}$$

Equations (5') (6') and (7') are easy to program, using little more than the multiplication of 2 x 2 matrices.

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Equations (2) and (3) become

$$\begin{bmatrix} f^{*}_{m} \\ \\ B^{*}_{m} \end{bmatrix} = \begin{bmatrix} \hat{f}_{m} \\ \\ \hat{B}_{m} \end{bmatrix} + h_{m}(B_{m}-\hat{B}_{m})\begin{bmatrix} \beta_{m} \\ \\ \delta_{m} \end{bmatrix} \tag{2'}$$

$$\begin{bmatrix} \hat{f}_{m} \\ \\ \hat{B}_{m} \end{bmatrix} = \begin{bmatrix} f^{*}_{m-1} \\ \\ B^{*}_{m-1} + f^{*}_{m-1} \Delta t \end{bmatrix} \tag{3'}$$

Hence, we find β_m and δ_m by iteration from equations (6') and (7'), starting with assumed values in (9) for a_0, b_0, c_0, d_0. Then use these values in equations (2') and (3') to compute f^{*}_m and B^{*}_m. We have elimi-nated equation (5') by writing out K(m) in equation (2').

RESULTS FOR KALMAN FILTERING

The results for our TMA No. 1 and TMA No. 2 when the bearing and bearing rate are computed by the Kalman filter are shown in Table 3. Also shown there are the results obtained by using pre-smoothing before applying the Kalman filter. The results obtained with groups of 2 are approxi-mately the same as those obtained with no pre-smoothing. When we tried groups of 5 or more the acceptance test in our TMA methods usually was not satisfied. Apparently, the Kalman filter requires a large number of effective observations to remove the errors that are introduced in the initial estimates of the covariance matrix (9). From these results it would appear that we should take frequent observations and not pre-smooth when using the Kalman filter.

The results obtained with the Kalman filter are comparable to the best results obtained with other methods of estimating the bearing and bear-ing rate.

We have not applied the Kalman filter to the direct estimation of the four parameters used in the CHURN TMA. Analyses of this type are re-ported in reference (d) for the linear filter and for the quadratic

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filter. The analysis is limited to the estimation of the parameters. It would appear to be necessary to repeat the entire computation to obtain the fire control orders and the miss distances to allow comparison with the results reported above.

A comparison shows that the above computations that are needed to apply Kalman filtering to our TMA methods are much simpler than the computations needed to apply it to the target position and velocity parameters, as described in reference (d). Our TMA methods are based on estimates of the bearing and bearing rate only, and these parameters are closely and simply related to the quantity being observed.

SUMMARY

Better results are obtained with our TMA methods when the bearing, as well as the bearing rate, is obtained by exponentially-weighted least squares, rather than using the final bearing as the estimator. When this procedure is used, pre-smoothing has little effect. However, the results are not significantly better than can be accomplished by splitting the bearing data on a tracking leg into two halves. Good results also are obtained with our TMA methods when Kalman filtering is used to estimate the bearing and bearing rate.

Some results obtained with our TMA methods using the best methods are shown below for $\sigma_b = 0.7$ degrees:

Course	$\hat{B}$, $\hat{\dot{B}}$	Group Size	TMA No. 1 $\overline{w}$	σ_w	TMA No. 2 $\overline{w}$	σ_w
90°	Exp. Sm.	1	-136	404	-146	380
		5	-110	442	-123	411
		25	-187	409	-203	381
	Kalman F.	1	-263	352	-272	331
-45°	Exp. Sm.	1	-109	473	-67	473
		5	- 4	569	39	564
		25	24	529	60	525
	Kalman F.	1	- 12	471	23	464

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With the CHURN TMA the best results (for pre-smoothing only) are obtained with groups of 10, for which the results are as follows:

Course	$\bar{w}$	σ_w
90°	512	283
-45°	-510	274

With the best available estimators of bearing and bearing rate our TMA methods yield significantly smaller biases in the miss distance than does CHURN with pre-smoothing, and somewhat larger standard deviations. Computations in reference (b) show that the bias usually has a larger effect than the standard deviation on the value of the acquisition probability. Computations in reference (a) show that the biases produced by delta biases and target maneuvers often are larger than the biases displayed above by large factors. Hence, a comparison of TMA methods depends on the effects produced by delta biases and target maneuvers to a greater extent than it depends on the effects produced by random bearing errors.

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TABLE 1. EFFECTS OF PRE-SMOOTHING BEARING DATA WITH OLD METHOD OF COMPUTING BEARING AND BEARING RATE

Target Course	Group Size	Number Rejects	TMA No. 1		TMA No. 2		CHURN	
			$\bar{w}$(yds)	σ_w(yds)	$\bar{w}$	σ_w	$\bar{w}$	σ_w
90°	1	15	384	973	401	927	2028	82
	5	4	-175	773	-205	774	789	182
	10	2	-292	555	-296	530	510	287
	25	0	-187	409	-203	381	409	861
-45°	1	22	-252	1150	-261	1119	-1089	47
	5	9	17	779	54	782	- 688	144
	10	2	147	864	181	855	- 522	283
	25	0	24	529	60	525	- 460	404

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TABLE II.   EFFECTS OF PRE-SMOOTHING BEARING DATA WITH
BEARING AND BEARING RATE BY EXPONENTIALLY-
WEIGHTED LEAST SQUARES

Target Course	Group Size	Number Rejects	TMA No. 1		TMA No. 2		CHURN	
			$\bar{w}$(yds)	σ_w (yds)	$\bar{w}$	σ_w	$\bar{w}$	σ_w
90°	1	0	-136	404	-146	380	2021	100
	5	2	-110	442	-123	411	801	175
	10	2	-107	570	-125	533	515	281
	25	0	-187	409	-203	381	409	861
-45°	1	0	-109	473	- 67	473	-1088	48
	5	2	- 4	569	39	564	- 620	180
	10	4	- 36	784	- 31	770	- 497	265
	25	0	24	529	60	525	- 460	404

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TABLE III. EFFECTS OF PRE-SMOOTHING BEARING DATA WITH BEARING AND BEARING RATE BY KALMAN FILTERING

Target Course	Group Size	Number Rejects	TMA No. 1		TMA No. 2	
			$\overline{w}$(yds)	σ_w(yds)	$\overline{w}$(yds)	σ_w(yds)
90°	1	0	-263	352	-272	331
	2	2	-280	357	-296	345
	5	Many	-	-	-	-
-45°	1	0	- 12	471	23	464
	2	6	11	453	44	448
	5	Many	-	-	-	-

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